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Using Geometry Expressions to Investigate Curves Drawn with Sticks and String

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Abstract: We all know that if you tie a string onto two sticks and draw keeping the string taut, you get an ellipse. Or do you? We investigate this question with sticks and string on an Oregon beach, and Geometry Expressions on a computer.

Introduction

In this paper we present a study of simple curves with Geometry Expressions. This is presented at a level which would allow it to be used as the basis for a lesson in high school.

Half an Ellipse

Attach a piece of string to two sticks on a beach and draw round with a third stick keeping the string tight.

Figure 1. The familiar construction for an ellipse

This can be modeled in Geometry Expressions directly. Let's assume that the string is length L and that the sticks are distance 2a apart. We set the locations of the sticks to be the points (-a,0) and (a,0). We then draw a triangle representing one position of the string. We make the distance to the apex from one of the sticks t (t will vary and be a parameter of the motion). We make the distance to the other stick L-t.

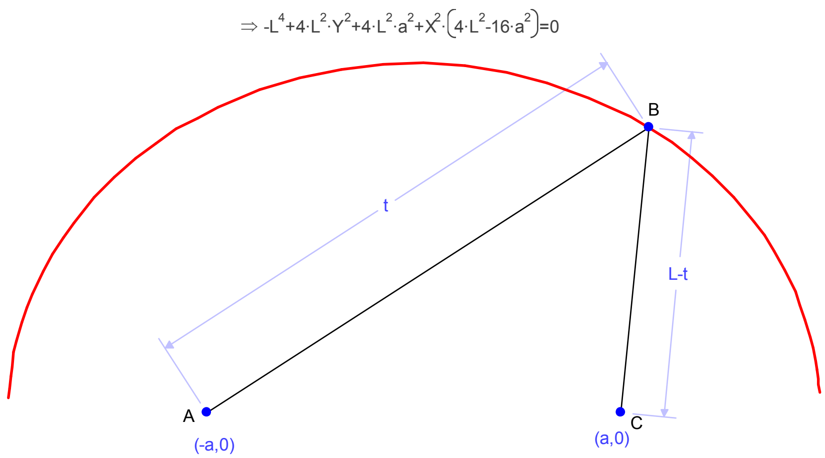


Figure 2: Geometry Expressions model of the ellipse construction

By examining the equation for the ellipse, can you determine the width and height of the ellipse? Can you verify these equations by thinking about the string?

Our software has only drawn half an ellipse. In the sand we have drawn half an ellipse.

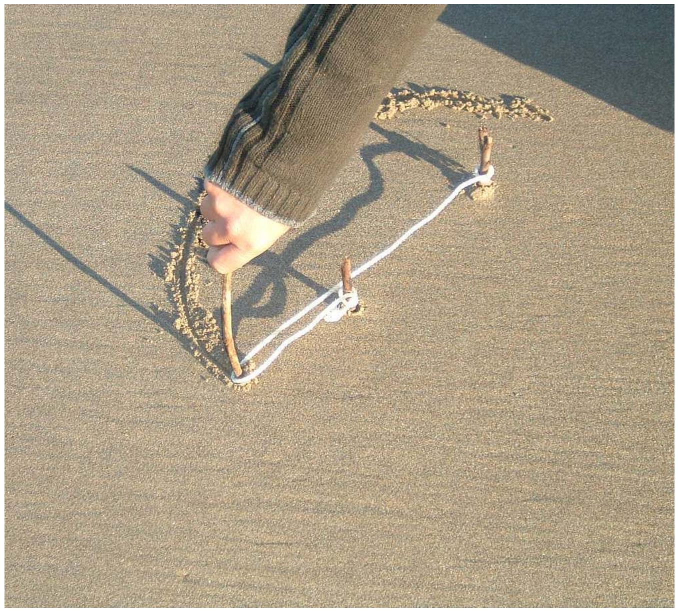


Figure 3: Completed semi-ellipse

What happens when you keep on going?

Beyond the Ellipse

Figure 4: String gets caught up on a stick

Keeping going, the string wraps around the stick. What shape is the next portion of the curve? Looking at the picture we can see that the curve is a portion of a circle, at least until we hit the string between the sticks. What is the radius of the circle?

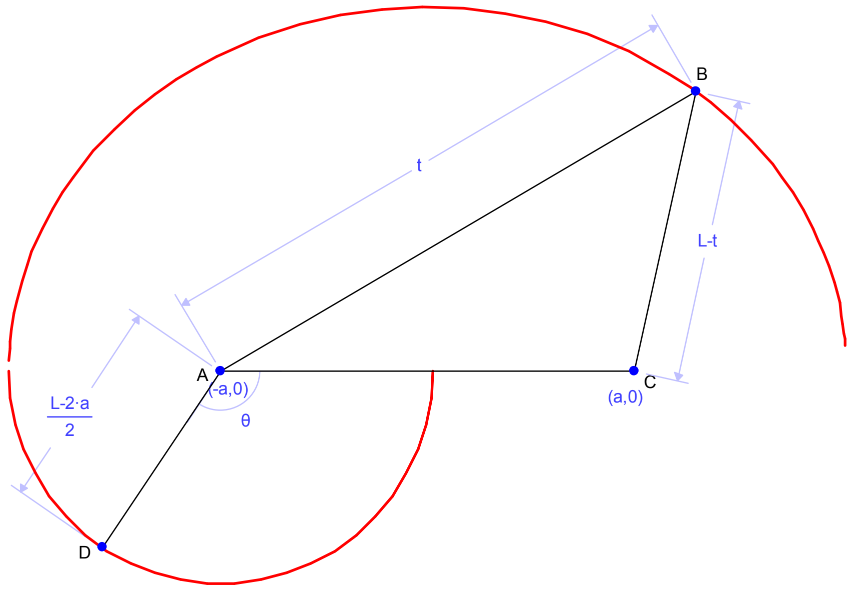


Figure 5: The second part of the curve is a semi circle

When we complete the semi circle, the string catches up with the taut string between the sticks and we trace another ellipse-like curve:

Figure 6: String interference

This can be modeled in Geometry Expressions by creating a new triangle and specifying that the distance from the first stick is s, and the distance from the second stick is L-3s.

$$\Rightarrow L^4+64\cdot X^4+128\cdot X^2\cdot Y^2+64\cdot Y^4+320\cdot X^3\cdot a+320\cdot X\cdot Y^2\cdot a-20\cdot L^2\cdot a^2+64\cdot a^4+Y^2\cdot\left(-20\cdot L^2+128\cdot a^2\right)+X^2\cdot\left(-20\cdot L^2+528\cdot a^2\right)+X\cdot\left(-32\cdot L^2\cdot a+320\cdot a^3\right)=0$$

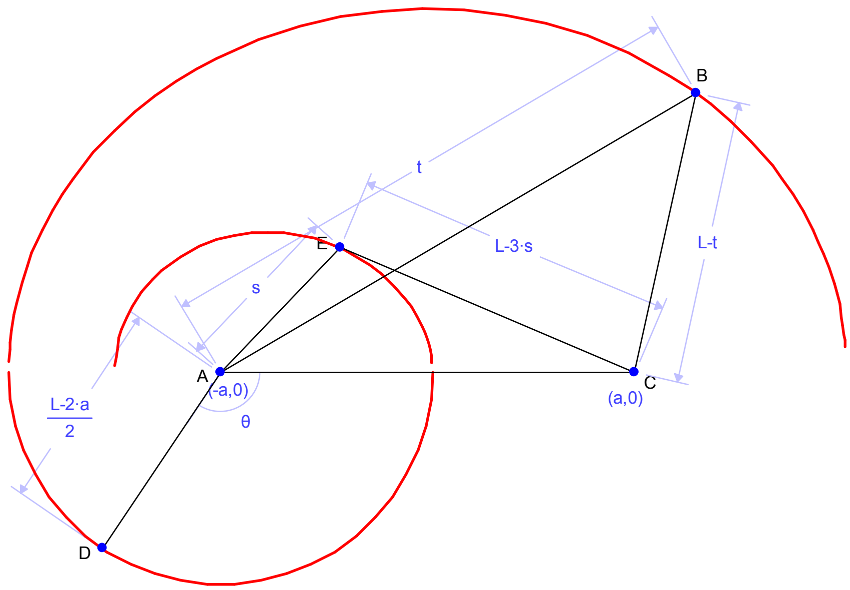


Figure 7: Modeling the next 180 degrees of the curve

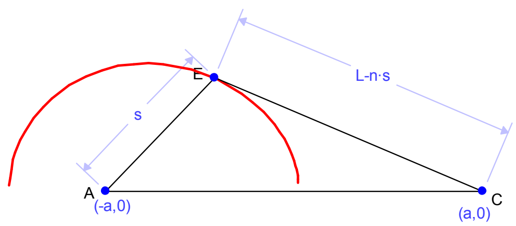
Is this curve actually part of an ellipse? What order is the curve?

Figure 8: The first 540 degrees of the curve

If we keep going with this process (difficult in real life, but easy on a computer), we can see that we will get a sequence of alternating semi-circles and other curves. What are the radii of the semi circles?

We can create the generic case of the upper curve by setting the distances in the above figure to s and L-n*s.

$$\Rightarrow L^4-2\cdot L^2\cdot a^2+a^4-2\cdot L^2\cdot a^2\cdot n^2-2\cdot a^4\cdot n^2+a^4\cdot n^4+X^4\cdot\left(1-2\cdot n^2+n^4\right)+Y^4\cdot\left(1-2\cdot n^2+n^4\right)+X^2\cdot Y^2\cdot\left(2-4\cdot n^2+2\cdot n^4\right)+Y^2\cdot\left(-2\cdot L^2+2\cdot a^2-2\cdot L^2\cdot n^2-4\cdot a^2\cdot n^2+2\cdot a^2\cdot n^4\right)+X^2\cdot\left(-2\cdot L^2+6\cdot a^2-2\cdot L^2\cdot n^2+4\cdot a^2\cdot n^2$$



The expression is complicated, but we can see that in general it is a 4th order curve (except of course when n=1 or -1, in which case all but the 2nd order terms vanish).